

# Nonclassicality of noisy quantum states

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**Abstract.** Nonclassicality conditions for an oscillator-like system interacting with a hot thermal bath are considered. Nonclassical properties of quantum states can be conserved up to a certain temperature threshold only. In this case, affection of the thermal noise can be compensated via transformation of an observable, which tests the nonclassicality (witness function). Possibilities for experimental implementations based on unbalanced homodyning are discussed. At the same time, we demonstrate that the scheme based on balanced homodyning cannot be improved for noisy states with proposed technique and should be applied directly.

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## 1. Introduction

Nonclassical properties play a crucial role in understanding fundamentals of quantum physics. This concept is the main theoretical background for many applications, including quantum information processing [1]. Usually, the nonclassicality manifests itself in specific properties of quantum statistics, which sometimes cannot be described in the framework of the probability theory [2].

This phenomenon was firstly considered in the famous work by Einstein, Podolsky, and Rosen [3] for demonstration of contradictions between quantum mechanics and the concept of “local realism”. As it was shown by Bell [4], the later leads to some inequalities violated for usual quantum mechanics. Experiments of Aspect with collaborators [5] confirmed this fact as well as assumption that quantum phenomena are characterized by a specific feature, nonlocality, which cannot be explained in classical terms. Note that nowadays there exist some other criteria for testing the presence of this kind of nonclassicality (entanglement) [6, 7, 8, 9].

Reconstruction of the Wigner probability distribution [10] in experiments with quantum tomography [11, 12, 13] demonstrates appearance of its negative values. This peculiarity cannot be explained in the framework of the probability theory, hence it does not have any classical counterparts. However, quadrature squeezing [14] as well as sub-Poissonian statistics of photon counts [15], being well-known examples of nonclassicality, is still possible for some states with completely positive Wigner function. This is explained by the fact that these features correspond to negative values of dispersions for some normally ordered observables, which are naturally determined via the Glauber-Sudarshan  $P$ -function [16, 17]. Properties of this distribution differ from those for the Wigner function. For example, there exist states characterized by both non-negative Wigner function and non-positive Glauber-Sudarshan  $P$ -function.

Therefore, the class of states characterized by non-positive  $P$ -function includes the states characterized by non-positive Wigner function, sub-Poissonian statistics of photon counts and quadrature squeezing. Hence, following to the works [18, 19], we will consider the nonclassicality as non-positivity of the  $P$ -function. Unlike the case of the Wigner function, this definition cannot be applied directly because of a very strong singularity of the  $P$ -function for nonclassical states.

The Bochner criterion [20] allows one to formulate observable conditions for experimental testing the nonclassicality [19]. Arguing in this way, we conclude that the  $P$ -function is a positive-definite one (or can be interpreted as probability density) if and only if for any function  $f(\alpha)$  the following inequality is satisfied:

$$\int_{-\infty}^{+\infty} d^2\alpha \int_{-\infty}^{+\infty} d^2\beta \Phi(\alpha - \beta) f(\alpha) f^*(\beta) \geq 0, \quad (1)$$

where  $\Phi(\beta)$  is the characteristic function of the  $P$ -distribution, *i.e.* its Fourier transform

$$\Phi(\beta) = \int_{-\infty}^{+\infty} d^2\alpha P(\alpha) \exp(\alpha^* \beta - \alpha \beta^*). \quad (2)$$

Hence, in order to test quantum states on the nonclassicality, it is sufficient to find such a function  $f(\alpha)$  that violates the inequality (1). An important example (as a matter of fact it was used in [19]) is the discrete variant of the Bochner criterion when this function is taken in the following form:

$$f(\alpha) = \sum_k \xi_k \delta(\alpha - \alpha_k), \quad (3)$$

where  $\xi_k$ ,  $\alpha_k$  are some arbitrary complex numbers. The experimental implementation of this criterion for optical fields was described in [21].

The inequality (1) can be rewritten in another equivalent form. For this purpose we introduce the object  $\mathcal{W}(\alpha)$ , which following [22] we will refer to as *the witness function* and define as following

$$\mathcal{W}(\alpha) = |g(\alpha)|^2 \geq 0, \quad (4)$$

where  $g(\alpha)$  is the Fourier image of  $f(\alpha)$ . It is easy to see that the inequality (1) takes now the following form:

$$\int_{-\infty}^{+\infty} d^2\alpha P(\alpha) \mathcal{W}(\alpha) \geq 0. \quad (5)$$

In other words, expression (5) means that the mean value of some operator  $\hat{\mathcal{W}}$  must be greater or equal to zero. This operator is defined in such a way that its normally ordered symbol is the witness function  $\mathcal{W}(\alpha)$ , *i.e.*

$$\mathcal{W}(\alpha) = \langle \alpha | \hat{\mathcal{W}} | \alpha \rangle, \quad (6)$$

where  $|\alpha\rangle$  is coherent state. Hence, if we succeed to find the operator  $\hat{\mathcal{W}}$  satisfying (6) and (4), such that its mean value is less than zero (or condition (5) fails to obey), then we can assert that the nonclassicality is inherent for the given state. As it was noted in [22], the concept of the witness function can be used for testing other kinds of nonclassicality as well.

The main problem for testing the nonclassicality for realistic systems is the decoherence [23]. It is reasoned by uncontrolled interaction of the system with an environment that leads to substantial effects on nonclassical properties. Depending on the system, the environment may have various physical nature. One of the well-known examples of such a system is an internal mode of high- $Q$  cavity [24, 25]. In this case, the number of external modes plays a role of the environment, which interacts with the system through the semitransparent mirror. Another example is absorption and scattering of the electromagnetic field by cavity mirrors while intracavity mode is extracted outside for the further use [26]. Absorption and scattering processes also take place while transferring a quantum electromagnetic signal through a semitransparent plate [27], waveguide *etc.*

The number of thermal photons in the optical domain of the electromagnetic radiation is negligibly small. Hence, in this case the environment can be regarded as being in the vacuum state. The main problem is that the modern technologies

in many cases do not afford to produce the optical devices with small interaction of the electromagnetic field and the absorbing medium. Especially this is apparent for optical high- $Q$  cavities, where resulting outgoing pulse includes just near 50% of the initial intracavity mode [26]. Somewhat the microwave cavities are devoid of this shortcoming. In this case, the constant of interaction between field and absorption system is comparatively small (see the discussion in [26]). However, the microwave domain is characterized by the presence of great number of thermal photons. This causes more serious difficulties in testing nonclassical properties of quantum states. Thus, there arises a natural question about a balance between the constant of interaction and temperature of environment for optimal detection of the nonclassicality. This is the subject of the presented paper.

The paper is organized as follows. In Sec. 2, we obtain an equation, which describes the evolution of the Glauber-Sudarshan  $P$ -distribution of an oscillator-like system under the thermal noise influence. In Sec. 3, we consider how to redefine the witness function and optimize testing the nonclassicality for noisy state. Application of this method with the schemes of unbalanced homodyne detection is considered in Sec. 4. In Sec. 5, we demonstrate that the discrete form of the Bochner criterion does not allow a further improvement and its presented form is the best choice even for the noisy states. An example of single-photon Fock number state is presented in Sec. 6. The last section contains the summary and conclusion.

## 2. Quantum state of system under the noise influence

We will describe open quantum systems using the input-output formalism, considered in the book [25]. It is worth noting that we focus on a class of linear systems only. It gives us a possibility of considering a wide enough class of experiments with quantum electromagnetic field of low intensity.

Following the input-output formalism, let us consider an open quantum system as a device with two input-output ports. One of them corresponds to a system and another one corresponds to the bath (see Fig. 1). Let the operator  $\hat{a}_{\text{in}}$  describes a system before interaction. For example, it can be an operator of the input-field mode. The operator  $\hat{c}_{\text{in}}$  describes degrees of freedom of an environment before interaction, *e.g.*, at the initial point of time. In the same manner we will describe the system after interaction in terms of the operator  $\hat{a}_{\text{out}}$ , which can be interpreted as the output-field mode operator. We also suppose that these operators satisfy the usual bosonic commutation relations

$$[\hat{a}_{\text{in}}, \hat{a}_{\text{in}}^\dagger] = 1, \quad (7)$$

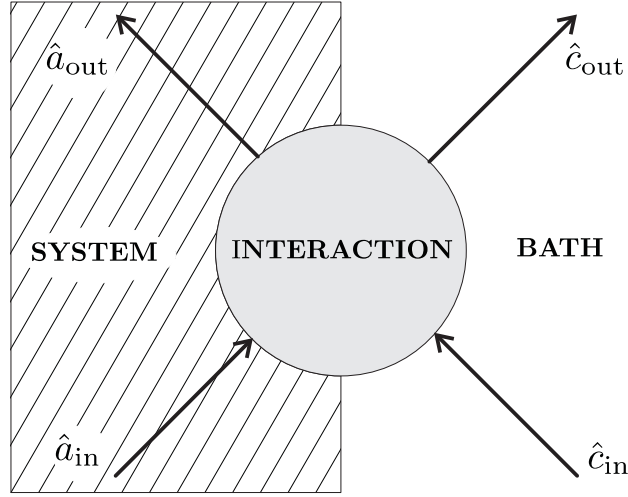
$$[\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger] = 1, \quad (8)$$

$$[\hat{c}_{\text{in}}, \hat{c}_{\text{in}}^\dagger] = 1. \quad (9)$$

Without loss of generality, we can describe the evolution of the system in terms of a linear input-output relation between these operators

$$\hat{a}_{\text{out}} = \sqrt{\eta} \hat{a}_{\text{in}} + \sqrt{1 - \eta} \hat{c}_{\text{in}}, \quad (10)$$

where  $0 \leq \eta \leq 1$  is efficiency of the input-output processes.



**Figure 1.** The model of open quantum system in terms of the input-output formalism.  $\hat{a}_{in}$  and  $\hat{c}_{in}$  are operators of the system and environment before interaction,  $\hat{a}_{out}$  and  $\hat{c}_{out}$  correspond to these operators after interaction.

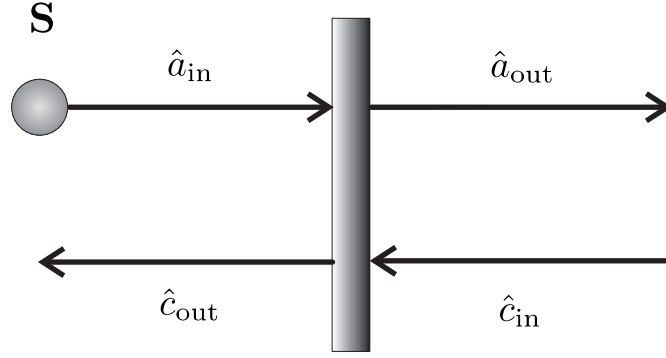
Let us regard some elementary examples. First of all consider transmission of a signal through a partially transparent plate (see Fig. 2). A source of quantum electromagnetic field  $S$  radiates the signal  $\hat{a}_{in}$  and, after transmitting through the plate, the output mode  $\hat{a}_{out}$  is detected. The bath system in this case is the electromagnetic-field modes  $(\hat{c}_{in}, \hat{c}_{out})$ , which pass in another direction. The plate plays a role of a device, which is responsible for a linear interaction between both the modes, *i.e.* between the system and the environment. Hence, Eq. (10) is the well-known input-output relation for the partially transparent plate to within a phase multiplier (see *e.g.* [28]). The efficiency  $\eta$  is connected with the transmission coefficient  $T$  of the plate in the following form:

$$\eta = |T|^2.$$

Another example is the process of quantum-state extraction from a high- $Q$  cavity [26]. In this case, the operator  $\hat{a}_{in}$  can be interpreted as the operator of an intracavity mode at the initial time. The operator  $\hat{a}_{out}$  corresponds to the non-monochromatic mode leaking from the cavity. The processes of absorption and scattering by mirrors can be considered as interaction between the system and the bath. Hence, the operator  $\hat{c}_{in}$  corresponds to the absorption system of the mirror and scattering modes of field. Corresponding input-output relation is considered in work [26]. The efficiency of this process is closely related to two components of the cavity decay rate:  $\gamma_{rad}$  which is responsible for the output and  $\gamma_{abs}$  which is responsible for the absorption and scattering

$$\eta = \frac{\gamma_{rad}}{\gamma_{rad} + \gamma_{abs}}. \quad (11)$$

As the next step we will rewrite the input-output relation (10) in the Schrödinger picture of motion, *i.e.* consider transformation of the density operator under the



**Figure 2.** Transmission of a signal through a partially transparent plate.  $\hat{a}_{\text{in}}$  and  $\hat{a}_{\text{out}}$  are input and output operators of the signal.  $\hat{c}_{\text{in}}$  and  $\hat{c}_{\text{out}}$  are operators of the field-mode passing in the inverse direction. Latter plays a role of the bath interacting with the signal (system) through a partially transparent plate.

noise influence. As it was already mentioned in the introduction, it is convenient to describe the nonclassicality using the Glauber-Sudarshan  $P$ -representation [16, 17]. The characteristic function (2) of the output field can be written down as follows:

$$\Phi_{\text{out}}(\beta) = \text{Tr} \left\{ \hat{\rho} e^{\hat{a}_{\text{out}}^\dagger \beta - \hat{a}_{\text{out}} \beta^* + \frac{|\beta|^2}{2}} \right\}, \quad (12)$$

where  $\hat{\rho}$  is density operator of the system and the bath. Moreover, we suppose that at the initial time it can be decomposed as

$$\hat{\rho} = \hat{\rho}_{\text{in}} \otimes \hat{\rho}_{\text{bath}}. \quad (13)$$

Substituting the input-output relations (10) into Eq. (12) and taking into account (13), one can obtain the following expression for the characteristic function of the output state:

$$\Phi_{\text{out}}(\beta) = \Phi_{\text{in}}(\beta\sqrt{\eta})\Phi_{\text{bath}}(\beta\sqrt{1-\eta}), \quad (14)$$

where  $\Phi_{\text{in}}(\beta)$  is the characteristic function of the input (noiseless) state of the system and  $\Phi_{\text{bath}}(\beta)$  is the characteristic function of the input state of the bath. We suppose that initially the bath is in the thermal state with temperature  $T$  and the mean value of thermal photons  $\bar{n} = \left[ \exp\left(\frac{\hbar\omega}{kT}\right) - 1 \right]^{-1}$ , i. e.,

$$\hat{\rho}_{\text{bath}} = \frac{1}{1 + \bar{n}} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^{\hat{n}}, \quad (15)$$

$$\Phi_{\text{bath}}(\beta) = e^{-|\beta|^2 \bar{n}}. \quad (16)$$

Hence, the characteristic function of the output (noisy) state is written in the following form

$$\Phi_{\text{out}}(\beta) = \Phi_{\text{in}}(\beta\sqrt{\eta}) e^{-|\beta|^2 \bar{n}(1-\eta)}. \quad (17)$$

One can easily see that for

$$\bar{n} = \frac{\eta}{1 - \eta} \quad (18)$$

the characteristic function (17) turns into a characteristic function for the positive-definite Husimi-Kano  $Q$ -distribution [29, 30] of a certain state. Thus, for such values of  $\bar{n}$  the  $P$ -function is always positive. One can say the same if the number of thermal photons is greater than the value determined by equation (18). Therefore, Eq. (18) defines *thermal threshold of the nonclassicality*. In other words, if the number of thermal photons in the bath is greater than the value given by Eq. (18), the nonclassicality in the sense of negative values of  $P$ -function always vanishes. However only the fact that the number of thermal photons is less than the thermal threshold defined by Eq. (18), cannot be considered as a sufficient condition of the nonclassicality.

Using the inverse Fourier transform we get from Eq. (17) the relation for the  $P$ -function of the state in the following form:

$$P_{\text{out}}(\alpha) = \frac{1}{\eta} \exp[\bar{n}(1-\eta)\Delta_\alpha] P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}\right), \quad (19)$$

where  $\Delta_\alpha = \frac{\partial^2}{\partial(\text{Re } \alpha)^2} + \frac{\partial^2}{\partial(\text{Im } \alpha)^2}$ . This means that the  $P$ -function for the output state satisfies the diffusion-like differential equation

$$\frac{\partial}{\partial \bar{n}} P_{\text{out}}(\alpha, \bar{n}) = (1-\eta)\Delta_\alpha P_{\text{out}}(\alpha, \bar{n}) \quad (20)$$

with the “initial” condition

$$P_{\text{out}}(\alpha, 0) = \frac{1}{\eta} P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}\right). \quad (21)$$

Therefore, taking into account above mentioned we can conclude that under the thermal noise influence the  $P$ -function of the system transforms according to the diffusion-like differential equation (20), where the mean number of the thermal photons plays the role of “time variable”. This means that with growing the number of thermal photons in the environment,  $P$ -function of the system is smoothed. For a certain value of  $\bar{n}$ , which is less or equal to the thermal threshold (18), domains of its negative values disappear. In [31] this critical number of thermal photons (in our notations this is  $\frac{1-\eta}{\eta}\bar{n}$ ) has been proposed to be used as a measure of nonclassicality.

### 3. Witness function for the output state

Let us suppose that a quantum state of the system already manifests the nonclassicality. In other words, according to Eq. (5), there exists such a witness function  $\mathcal{W}(\alpha)$  that

$$\bar{\mathcal{W}} = \int_{-\infty}^{+\infty} d^2\alpha P_{\text{in}}(\alpha) \mathcal{W}(\alpha) < 0. \quad (22)$$

Consider corresponding state of the output mode given by Eq. (19). To order it manifests the property of the nonclassicality as well, another witness function  $\mathcal{W}_{\text{th}}(\alpha, \bar{n})$  should exist and the following inequality should be true:

$$\bar{\mathcal{W}}_{\text{th}} = \int_{-\infty}^{+\infty} d^2\alpha P_{\text{out}}(\alpha) \mathcal{W}_{\text{th}}(\alpha, \bar{n}) < 0. \quad (23)$$

We now want to check whether  $\mathcal{W}_{\text{th}}(\alpha, \bar{n})$  can be chosen in such a form that  $\bar{\mathcal{W}} = \bar{\mathcal{W}}_{\text{th}} < 0$ . For doing this let us substitute Eq. (19) for the  $P$ -function of the output state into Eq. (23)

$$\bar{\mathcal{W}}_{\text{th}} = \int_{-\infty}^{+\infty} d^2\alpha \left[ \frac{1}{\eta} \exp[\bar{n}(1-\eta) \Delta_\alpha] P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}\right) \right] \mathcal{W}_{\text{th}}(\alpha, \bar{n}). \quad (24)$$

Taking into account that the diffusion operator is a Hermitian one and penetrating the change of variables in the last equation we can rewrite it as follows:

$$\bar{\mathcal{W}}_{\text{th}} = \int_{-\infty}^{+\infty} d^2\alpha P_{\text{in}}(\alpha) \left[ \exp\left(\frac{\bar{n}(1-\eta)}{\eta} \Delta_\alpha\right) \mathcal{W}_{\text{th}}(\alpha\sqrt{\eta}, \bar{n}) \right]. \quad (25)$$

Comparing Eqs. (22), (25), one can conclude that  $\bar{\mathcal{W}} = \bar{\mathcal{W}}_{\text{th}}$  if

$$\mathcal{W}(\alpha) = \exp\left[\frac{\bar{n}(1-\eta)}{\eta} \Delta_\alpha\right] \mathcal{W}_{\text{th}}(\alpha\sqrt{\eta}, \bar{n}). \quad (26)$$

Inverting this equation, one obtains the following expression for the witness function, which tests the nonclassicality for the output (noisy) state

$$\mathcal{W}_{\text{th}}(\alpha, \bar{n}) = \exp[-\bar{n}(1-\eta) \Delta_\alpha] \mathcal{W}\left(\frac{\alpha}{\sqrt{\eta}}\right). \quad (27)$$

This means that we can find the new witness function as a solution of diffusion-like differential equation with a negative diffusion coefficient

$$\frac{\partial}{\partial \bar{n}} \mathcal{W}_{\text{th}}(\alpha, \bar{n}) = -(1-\eta) \Delta_\alpha \mathcal{W}_{\text{th}}(\alpha, \bar{n}), \quad (28)$$

and the following “initial” condition

$$\mathcal{W}_{\text{th}}(\alpha, 0) = \mathcal{W}\left(\frac{\alpha}{\sqrt{\eta}}\right). \quad (29)$$

Therefore, for a noisy state the witness function can be redefined in such a way that testing the nonclassicality gives a result equal to the noiseless case. The fact that Eq. (28) is a diffusion-like equation with negative diffusion coefficient means that with growing the number of thermal photons  $\bar{n}$  one has to choose a sharper witness function for obtaining the same result. It is clear that this possibility exists up to a certain temperature threshold only, which however can be less than the value defined by Eq. (18) but cannot be greater. Beyond it the solution of Eq. (28) may have both strong singularities and domains of negative values.

#### 4. Testing the nonclassicality in experiments with unbalanced homodyning

Unbalanced homodyning, proposed in [32], allows one to test the nonclassicality with an important class of witness functions, which have a form of the Gauss distribution

$$\mathcal{W}(\alpha) = \frac{1}{\pi a^2} \exp\left[-\frac{|\alpha - \gamma|^2}{a^2}\right]. \quad (30)$$



Firstly consider the case of a noiseless quantum state, *i.e.* the input field. Inserting the witness function (30) into Eq. (22) we get that  $\bar{\mathcal{W}}$  can be regarded as value of some  $s$ -parameterized distribution [33] in point  $\gamma$

$$\bar{\mathcal{W}} = \frac{1}{\pi a^2} \int_{-\infty}^{+\infty} d^2\alpha P_{\text{in}}(\alpha) \exp \left[ -\frac{|\alpha - \gamma|^2}{a^2} \right] = P_{\text{in}}(\gamma, s), \quad (31)$$

where

$$s = 1 - 2a^2. \quad (32)$$

It is clear that it never tests the nonclassicality for  $s \leq -1$ , *i.e.*  $a^2 \geq 1$ .

The scheme of the corresponding experiment is presented in Fig. 3. The signal, which is tested for the nonclassicality, is combined through a beam splitter with the local-oscillator field. The photon-counting detector, placed in an output port, is used for the measurement of counting distributions  $P_n^{\text{in}}(\gamma, \eta_{\text{h}})$ . They depend on the coherent amplitude  $\gamma$ , and the overall efficiency of the homodyning  $\eta_{\text{h}}$ , which are expressed in terms of amplitude of the local oscillator  $\gamma_{\text{lo}}$ , transmission  $T$  and reflection coefficients  $R$  of the beam splitter and the efficiency of photon counting  $\zeta$

$$\gamma = -\frac{R}{T} \gamma_{\text{lo}}, \quad (33)$$

$$\eta_{\text{h}} = \zeta |T|^2. \quad (34)$$

As it was shown in [32] the value of  $\bar{\mathcal{W}}$ , given by Eq. (31), can be reconstructed from the probabilities of photon counts using the following expression

$$\bar{\mathcal{W}} = \frac{1}{\pi a^2} \sum_{n=0}^{+\infty} \left[ -\frac{1 - \eta_{\text{h}} a^2}{\eta_{\text{h}} a^2} \right]^n P_n^{\text{in}}(\gamma, \eta_{\text{h}}). \quad (35)$$

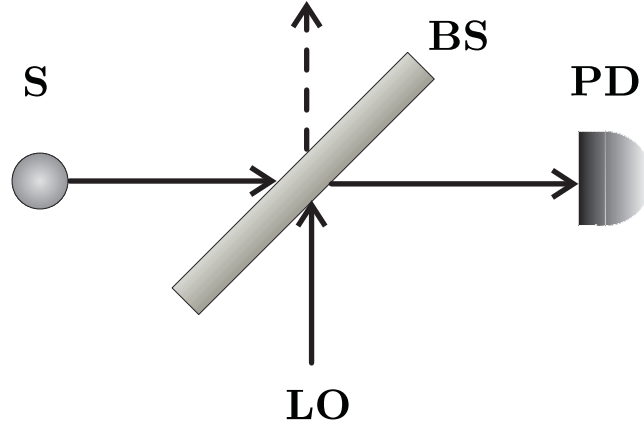
Assume that the nonclassicality can be detected for the noiseless signal with the above described procedure. This gives us a possibility to find such a witness function, which tests the nonclassicality for the corresponding noisy state. It can be obtained by resolving the diffusion-like equation (28), with ‘initial’ condition (29) specified by function (30). The solution is written as follows:

$$\mathcal{W}_{\text{th}}(\alpha) = \frac{1}{\pi \left( a^2 - \bar{n} \frac{\eta}{1-\eta} \right)} \exp \left[ -\frac{\left| \frac{\alpha}{\sqrt{\eta}} - \gamma \right|^2}{a^2 - \bar{n} \frac{\eta}{1-\eta}} \right]. \quad (36)$$

It is worth noting that this solution is defined just for

$$\bar{n} \leq a^2 \frac{\eta}{1-\eta}. \quad (37)$$

For other values of  $\bar{n}$ , it is not positive definite and has strong singularities. This is a typical property for the solution of diffusion equation with negative diffusion coefficient. The last expression defines the thermal threshold for the scheme of unbalanced homodyning. Taking into account that the maximal value of  $a^2$  is 1, one immediately obtains Eq. (18).



**Figure 3.** The scheme of unbalanced homodyne detection.

Corresponding value of  $\bar{\mathcal{W}}_{\text{th}}$  as well as  $s$ -parameterized distribution for the noiseless signal can be reconstructed from the probabilities of photon counts for noisy signal  $P_n^{\text{out}}(\gamma, \eta_h)$  using the following expression:

$$\begin{aligned} \bar{\mathcal{W}}_{\text{th}} &= P_{\text{in}}(\gamma, s) \\ &= \frac{\eta}{\pi \left( a^2 - \bar{n} \frac{\eta}{1-\eta} \right)} \sum_{n=0}^{+\infty} \left[ -\frac{1 - \eta_h \left( a^2 - \bar{n} \frac{\eta}{1-\eta} \right)}{\eta_h \left( a^2 - \bar{n} \frac{\eta}{1-\eta} \right)} \right]^n P_n^{\text{out}}(\gamma \sqrt{\eta}, \eta_h), \end{aligned} \quad (38)$$

where  $s$  is defined by Eq. (32). A disadvantage of this method is a fact that for some quantum states the series defined by Eqs. (35), (38) may diverge (see [32]). This gives some restrictions for the application of this method.

## 5. Discrete form of the Bochner criterion

Discrete form of the Bochner criterion allows one to formulate observable conditions [19] based on the technique of balanced homodyning. Combining Eqs. (1) and (3), one obtains the following inequality:

$$\sum_{k,l} \Phi_{\text{in}}(\alpha_k - \alpha_l) \xi_k \xi_l^* \geq 0. \quad (39)$$

In case if it violates for some values of parameters (*i.e.*, the quadratic form is not a positive-definite one), the state has nonclassical properties. In other words, the nonclassicality appears if at least one minor of the matrix  $\Phi_{l,k}^{\text{in}} = \Phi_{\text{in}}(\alpha_k - \alpha_l)$  has a negative value for certain values of  $\alpha_k$ . This matrix depends just on the values of characteristic function for the  $P$ -distribution in certain points. The latter can be reconstructed via measurements of quadratures. For the electromagnetic fields, this can be performed with procedure of the balanced homodyne detection [34]. Corresponding methods are well known for other systems [13].

The witness function  $\mathcal{W}(\alpha)$  for the discrete form of the Bochner criterion can be easily found utilizing the definition given by Eq. (4)

$$\mathcal{W}(\alpha) = \sum_{k,l} \xi_k \xi_l^* \exp [\alpha^* (\alpha_k - \alpha_l) - \alpha (\alpha_k^* - \alpha_l^*)]. \quad (40)$$

Now we may consider a question whether it is possible to find the witness function  $\mathcal{W}_{\text{th}}(\alpha)$  which tests the nonclassicality for a noisy state in the same manner as for the corresponding noiseless one. For this purpose, we should resolve the diffusion-like equation (28), with “initial” condition (29) specified by the function (40). The result is written in the following form:

$$\mathcal{W}_{\text{th}}(\alpha) = \sum_{k,l} \xi_k \xi_l^* e^{\bar{n} \frac{1-\eta}{\eta} |\alpha_k - \alpha_l|^2} \exp \left[ \alpha^* \frac{\alpha_k - \alpha_l}{\sqrt{\eta}} - \alpha \frac{\alpha_k^* - \alpha_l^*}{\sqrt{\eta}} \right]. \quad (41)$$

This function is not a positive-definite one and consequently it can not be considered as a witness function. Therefore, the witness function in the form of Eq. (40) cannot be improved with described method. Hence, the nonclassicality conditions in form of the discrete Bochner criterion [19] should be applied directly which is the best choice even for the noisy state.

## 6. An example: Fock state

Single-photon Fock number state with density operator

$$\hat{\rho}_{\text{in}} = |1\rangle\langle 1|. \quad (42)$$

is a good candidate for the experimental realization of the proposed method. Generation of this state with using the frequency down-conversion process and testing it for the nonclassicality with the application of the balanced homodyne detection is reported in [21].  $s$ -parameterized distribution for the single-photon Fock state has the following form:

$$P_{\text{in}}(\alpha, s) = \begin{cases} \frac{2}{\pi(1-s)^3} (4|\alpha|^2 - 1 + s^2) \exp \left[ -\frac{2|\alpha|^2}{1-s} \right], & -1 \leq s < 1 \\ \left( 1 + \frac{1}{4} \partial_{\alpha^* \alpha}^2 \right) \delta(\alpha), & s = 1 \end{cases}, \quad (43)$$

This is a regular function for  $-1 \leq s < 1$ , and a distribution with very strong singularity for  $s = 1$ , *i.e.* for the Glauber-Sudarshan  $P$ -function. It is clear that this state has nonclassical properties and non-positive-definite phase-space distributions for all values of the parameter  $s \neq -1$ .

Different losses in experimental set-up lead to admixing the vacuum state into the density operator (42). Hence, the resulting state has the form of the following statistical mixture:

$$\hat{\rho}_{\text{out}} = \eta |1\rangle\langle 1| + (1 - \eta) |0\rangle\langle 0|. \quad (44)$$

This is a result of interaction between the field mode and zero-temperature bath (where the mean number of thermal photons  $\bar{n}$  is negligible). As it was shown in [21], the nonclassicality can be tested, at least in principle, for any value of the efficiency  $\eta$ .

Such situation is usual for the optical domain. We consider a more general case, when the bath does have non-zero temperature, that is typical for the microwave domain, vibrational motion of trapped atom, *etc.*

The Glauber-Sudarshan  $P$ -function for the thermal noisy state can be obtained from Eq. (19) and is written as follows:

$$P_{\text{out}}(\alpha) = \frac{1}{\eta} P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}, s'\right), \quad (45)$$

where

$$s' = 1 - 2\bar{n}\frac{1-\eta}{\eta}. \quad (46)$$

The right-hand side of this equation in the case of single-photon Fock state has non-positive values for  $s' > -1$ . Therefore, taking into account Eq. (46) we conclude that this state under the thermal noise influence preserves nonclassical properties up to the thermal threshold given by Eq. (18).

Application of the unbalanced homodyning scheme, considered in Sec. 4, means testing the nonclassicality with the witness function  $\mathcal{W}(\alpha)$  given by Eq. (30). Hence, according to Eq. (31) for single-photon (noiseless) Fock state one has the following value for the quantity  $\bar{\mathcal{W}}$ :

$$\bar{\mathcal{W}} = P_{\text{in}}(\gamma, 1 - 2a^2) = \frac{1}{\pi a^6} (|\gamma|^2 + a^2(a^2 - 1)) \exp\left[-\frac{|\gamma|^2}{a^2}\right]. \quad (47)$$

It has negative values for any  $a^2 < 1$  and, moreover, shows non-positivity of the phase-space distribution with  $s = 1 - 2a^2$ .

Applying the same witness function for single-photon Fock state under thermal noise influence, one obtains the following value:

$$\bar{\mathcal{W}}' = \frac{1}{\pi a^2} \int_{-\infty}^{+\infty} d^2\alpha P_{\text{out}}(\alpha) \mathcal{W}(\alpha) = \frac{1}{\eta} P_{\text{in}}\left(\frac{\gamma}{\sqrt{\eta}}, s'\right), \quad (48)$$

where

$$s' = 1 - 2\frac{\bar{n}(1-\eta) + a^2}{\eta}. \quad (49)$$

This procedure can test the nonclassicality only if  $s' > -1$ . In other words, testing the nonclassicality with the witness function (30) is impossible if  $a^2 \geq \eta - \bar{n}(1-\eta)$ .

However, in case when the mean number of thermal photons  $\bar{n}$  is less than the thermal threshold, one can apply the witness function  $\mathcal{W}_{\text{th}}(\alpha)$  that is given by Eq. (36). In an experiment, the value of  $\bar{\mathcal{W}}_{\text{th}}$  can be reconstructed with using Eq. (38). This gives a numerical result, which is equal to Eq. (47), that indicates both the presence of nonclassicality and non-positive values for the phase-space distribution with  $s = 1 - 2(a^2\eta - \bar{n}(1-\eta))$ .

## 7. Conclusions

Interaction of the system with an environment leads to the disappearance of its nonclassical properties. Mostly this process is known as decoherence, when off-diagonal matrix elements of the density operator vanish in a certain representation. As a result, initially pure quantum state evolves to a mixed one.

The phenomenon of nonclassicality has different manifestations such as sub-Poissonian statistic of photon counts, photon antibunching, quadrature squeezing, *etc.* Following to [19] we consider the state as nonclassical one if its Glauber-Sudarshan  $P$ -function cannot be interpreted as probability distribution. Indeed, for the specific quantum states this function has negative values and, moreover, has very strong singularities, which do not allow us to express it in terms of regular functions.

If the environment has a non-zero temperature, the corresponding system can be considered as being under the thermal noise influence. In this case, the Glauber-Sudarshan  $P$ -function smoothes, and domains of negative values as well as singularities disappear. Eventually, beyond a certain value of the temperature (thermal threshold), the system does not manifest any nonclassical properties.

We consider the case of an oscillator-like system (*e.g.* a mode of the electromagnetic field) interacting with hot environment of other oscillators (other modes of field, absorption system, *etc.*). The Glauber-Sudarshan  $P$ -function of such a system under the noise influence evolves according to the diffusion-like equation, where the mean number of photons  $\bar{n}$  plays a role of “the time variable” as well as “diffusion coefficient” is expressed in terms of the corresponding efficiency.

The most convenient method for testing the nonclassicality is the measurement of a certain observable. Its normally-ordered symbol is called the witness function. Negative mean value of this observable indicates nonclassical properties of the corresponding quantum state. In principle, the concept of the witness function is quite general and can be used for testing other forms of the nonclassicality [22].

In case if the temperature is less than the thermal threshold and noiseless state is tested for the nonclassicality with the witness function  $\mathcal{W}(\alpha)$ , then there exists (but not always) other witness function  $\mathcal{W}_{\text{th}}(\alpha)$ , which has the same mean value for the noisy state as  $\mathcal{W}(\alpha)$  for the noiseless one. This new witness function can be obtained as a solution of a diffusion-like equation with negative diffusion coefficient. This feature explains both the existence of the thermal threshold of the nonclassicality and restrictions for the application of the proposed method. Indeed, if the mean number of thermal photons  $\bar{n}$ , that plays a role of “time variable” in this equation, is greater than a certain value, then corresponding solution may not be a positive-definite one and, moreover, has strong singularities.

An example is the witness function chosen in a form of the Gauss distribution with dispersion  $a^2$ . The corresponding mean, that is a value of the phase space distribution with  $s = 1 - 2a^2$  in a certain point, can be reconstructed in an experiment using procedure of unbalanced homodyne detection [32]. The evolution of this witness function

according to the diffusion-like equation with negative diffusion coefficient results in decreasing of the dispersion. It is clear that there exists a value of  $\bar{n}$  when it degenerates into the  $\delta$ -function. Beyond this value, the solution is defined in the space of distributions which, moreover, are not positive-definite ones. Hence, testing the nonclassicality beyond a certain threshold is impossible.

Another method for testing the nonclassicality is the application of the discrete form of the Bochner criterion. This technique requires to know values of the characteristic function in some points. It can be realized via measurements of quadratures. Corresponding procedures are developed for different systems [13]. Particularly, for the one-mode electromagnetic field it can be performed with balanced homodyne detection. However, in this case utilizing the diffusion-like equation with negative diffusion coefficient, one obtains the witness function  $\mathcal{W}_{\text{th}}(\alpha)$ , which is not positive-definite one for any  $\bar{n} > 0$ . This means that this technique cannot be improved and the discrete Bochner criterion should be applied directly.

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